Towards a phase diagram of the 2D Skyrme model

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PACS. 12.39.Dc – Skyrmions.
PACS. 73.43.-f – Quantum Hall effects.
PACS. 05.10.-a – Computational methods in statistical physics and nonlinear dynamics.

Abstract. – We discuss calculations of the phase diagram of the baby-Skyrme model, a two-dimensional version of the model that has been so successful in the description of baryons. Contact is made with the sine Gordon model in 1D, and relations with the Skyrme model used in the quantum-Hall effect are pointed out. It is shown that at finite temperature the phase diagram is dominated by a liquid, and not the crystal that plays a role for zero temperature.

Introduction. – The Skyrme model has a venerable history [1] in the non-perturbative description of nucleon structure and the low-energy behaviour of baryonic matter, since it contains a good description of the long-wave length behaviour of the dynamics of hadrons. As has been argued by 't Hooft and Witten [2], this is closely related to the large-number-of-colours limit of QCD, in which baryons must emerge as solitons, in much the same way as happens in the Skyrme model. Alternatively, we can interpret the model in terms of chiral perturbation theory, which corresponds closely to a gradient expansion in terms of the pion field. The model can be used to describe, with due care [3], systems of a few nucleons, and has also been applied to nuclear and quark matter. One of the most complicated aspects of the physics of hadrons is the behaviour of the phase diagram of hadronic matter at finite density, and low or even zero temperature. The complex interplay between confinement and chiral symmetry is difficult to describe, and every model has its own draws-backs. Within the standard zero-temperature Skyrme model description there are signatures of chiral symmetry restoration at finite density, but in a rather strange way, where a crystal of nucleons turns into a crystal of half nucleons at finite density, which is chirally symmetric only on average [4]. The question of the finite-temperature behaviour has never been addressed and would be of some interest. Since the behaviour of nuclear matter at high densities is expected to reveal, probably by one or two phase transitions, the substructure of baryonic matter, it is interesting to study this phenomenon in Skyrme models. Before doing so, we would like to study a slightly simpler form of the model, by reducing the number of dimensions to two.

The two-dimensional Skyrme model also has physical relevance; a special form of the model is used for ferromagnetic quantum Hall (QHF) systems [5]. This effective theory is obtained when the excitations relative to the $\nu = 1$ ferromagnetic quantum Hall state are described

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in terms of (a gradient expansion in) the spin density, a field with properties analogous to the pion field in nuclear physics [6]. Apart from obvious changes due to the number of dimensions, the new approach differs from the historical Skyrme model by having a different time-dependent term in the Lagrangian, and the appearance of a non-local interaction, where the topological charge density at different points interacts through the Coulomb force. In the limit of large Skyrmions this last term can be approximated by a more traditional local "Skyrme term", which is quartic in the fields, leading to the standard baby-Skyrme model with local interactions. Even with the non-local complications the model is on the whole remarkably similar to the nuclear Skyrme model. The Skyrmionic effective degrees of freedom describe the ground state of such systems, and probably also the low energy dynamics and thermodynamics, so that we can ask similar questions as for baryonic matter.

It has been argued [7] that for filling factors not too far from $\nu = 1$ the quantum ground state is described by a crystalline solution of the Skyrme model, even though no direct experimental proof exists. The most exciting possible experimental signature is a low-temperature phase transition, which may be associated with Skyrmion-lattice melting [8]. This last paper also makes an important study of the possible phases of the quantum Hall Skyrme model. Like in most other work on the subject [9, 10] these calculations are based on an assumption that the Skyrmions in a crystal are identical to free Skyrmions, which is at best a crude approximation of the real situation. At the same time it allows us to look at the phase diagram of quantum Hall Skyrmionic systems, which may well be the easiest way to calculate parts of the phase diagram of the underlying electronic model.

In this paper we shall consider the local baby Skyrme model, which can be seen as a dimensional reduction of the 3D Skyrme model, or as the local limit of the QHF Skyrme model. It is also related to the sine-Gordon model in 1D, by a simple dimensional reduction. Thus all numerical techniques can first be tested on this simple and exactly solvable model. All of these models contain a topologically conserved charge, which gives rise to topological solitons. In 3D (nuclear physics) these are identified as baryons, and in QHF systems as quasi-particles.

At finite density but zero temperature the Skyrme models and the sine Gordon model have a crystalline structure, which consist of regularly spaced solitons. In the one-dimensional case, where exact solutions exist [11], this is known to be a zero temperature artifact, and we have a liquid at any finite temperature, no matter how small. In the nuclear Skyrme model we would like to find a fluid, to mimic the quantum liquid behaviour expected in nuclear matter [12]. In order to appreciate the subtleties involved, one must understand that the Skyrme model, as a classical field theory, can be understood as a semiclassical (large action) limit of a quantum theory. Clearly the quantum fluctuations in the underlying theory could be large enough to wash out the crystalline structure, as happens for the sine Gordon model. In the special case of one space dimension it is also well known that thermal and quantum fluctuations play exactly the same role, and thermal fluctuations also break the crystalline state. For the Skyrme and baby Skyrme models it is not easy to access the quantum fluctuations, since the field theories are non-renormalisable, but we can access the thermal fluctuations (with care, because nonremormalisability plays a role in the thermodynamics as well!). Furthermore, we might well wish to study the physics of these systems at temperatures where thermal fluctuations dominate the physics. Therefore, we shall concentrate on the finite temperature phases of these classical field theories.

Theoretical and computational background. – The most direct approach to the problem is to perform a Monte-Carlo (Metropolis algorithm based) study of the partition function of each of these models. As we shall argue below the least obvious aspect in such an approach is

how to deal with the topological conservation laws. Rather than immediately tackle the 3D model, we concentrate on the (local) 2D model as a first example, for which the visualization of, and thus the understanding gained from, the results is much more straightforward. The extensions to 3D (nuclear physics) and quantum-Hall Skyrmions are under way and will be presented elsewhere [13,14]. We expect the current for the QHE solitons to be very similar to the results reported here.

The model is defined by the Lagrange density, which in D dimensions takes the form (we shall work in "natural" units in which the parameters α and β equal 1)

$$\mathcal{L} \;=\; \frac{\alpha}{2}\sum_{k,\mu}(\partial_\mu\phi_k)^2 - \frac{\beta}{4}\left(\sum_{k,\mu}(\partial_\mu\phi_k)^2\right)^2 - \frac{1}{16}\sum_{k,l}\Big(\sum_{\mu}\partial_\mu\phi_k\partial_\mu\phi_l\Big)^2 - m_\pi^2(1-\phi_1), \quad (1)$$

where the coordinates μ run from 0 (time) to D, and the vector field ϕ has $D+1$ components, and has unit length

$$\sum_k \phi_k^2 = 1. \quad (2)$$

This form of Lagrangian is the exact analogue of the original nuclear (3D) one, apart from the "mass term" which was first added to the 3D model by Adkins *et al* [15] to improve the physics. In the 2D model it is crucial for the stability of the solitonic solutions; in the QHE eefct such a term natural arrises from the Zeeman effect. Nonetheless the current model differs in several important aspects from the quantum-Hall Skyrme model, the main differences are the fourth-order term, which is non-local in the QHE, and the second order time derivatives, where only a first order-one appears in the Lagrangian of the QHE (see, however, Ref. [16]). Nonetheless, it might be expected that the current model shares many of the characteristics of the 3D model and the QHE Skyrmions, and at least the zero-temperature results are independent of the term containing time-derivatives.

Apart from the question what the few-soliton solutions of the model look like [17], one might ask what phases are shown by the model at finite density and temperature. The way we shall address this problem is by using a Metropolis approach, where the partition function is sampled using a pseudo-time dynamics. We first discretize space on a lattice, and then calculate the partition function,

$$Z_N(\beta) = \int \prod_i d^3\phi_i\,\delta(\vec{\phi}_i^{\,2}-1)\,d^3\dot{\phi}_i\,\delta(\dot{\vec{\phi}}_i\cdot\vec{\phi}_i)\exp(-\beta E), \quad (3)$$

where i runs over all lattice sites. Here E is the energy expression corresponding to the Lagrangian (1), and the delta functions arise from the unitarity constraint and its time derivative. Since we cannot work with an infinite system, we use a finite box with periodic boundary conditions. This then raises the final question, why we have chosen to use a canonical (rather than grand-canonical) ensemble in the expression above. The reason for this choice combines practical limitations with topological ones: We have found it impossible to design a sufficiently simple (i.e., localised) update mechanism in the Metropolis algorithm that succeeds in generating additional topological charge. This is clearly related to the non-local nature of the topological conservation laws, and we find that the charge is essentially constant over the whole Monte-Carlo iteration, even if we add an additional chemical potential term μB to the exponential in Eq. (3). Since the total charge is thus fixed by the initial one, our approach is rather ineffective in sampling the grand-canonical ensemble. In order to sample this fully, we

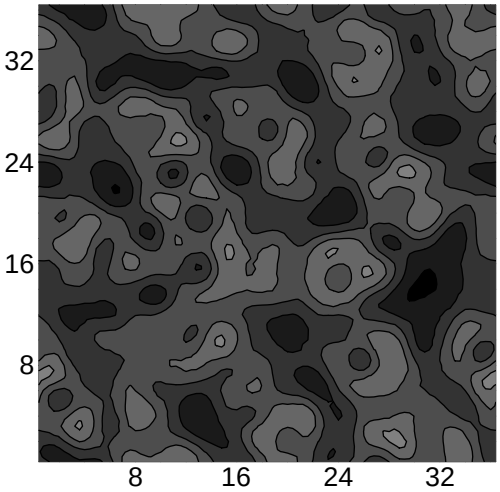

Fig. 1

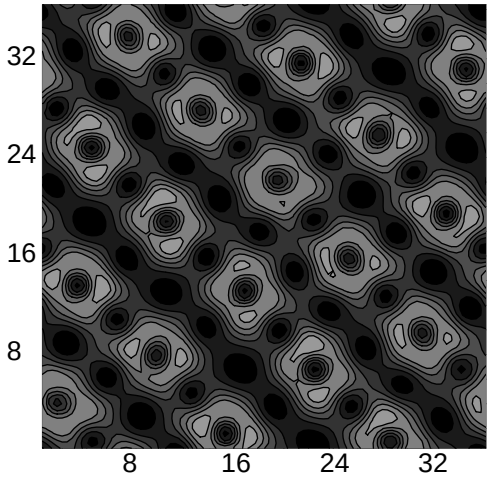

Fig. 2

Fig. 1 – The average of the baryon density for a fluid phase. The number of simulations included in the average was chosen small, do that some structure is still visible.

Fig. 2 – The ensemble average of the baryon density for a solid.

must superimpose the results from various canonical simulations with different initial baryon number. Alternatively we have found it convenient to study systems with open boundaries, an approach not pursued in this work. Such an approach may show even more promise when combined with a heat bath attached to those boundaries, or related Langevin calculations, see e.g. [18].

Results. – The Monte-Carlo simulations were verified by comparison with exact results for the sine-Gordon model [11], and found to be quite accurate, up to truncation errors due to the lattice spacing. Full results shall be presented in Ref. [13].

Our main goal is to understand the thermodynamics, and the order in the system as a function of density ρ and temperature T. To that end we need to be able to distinguish fluid phases from solid ones. The most visual way is the pseudo-time averaged topological density. For a fluid, which its large mobility, such an average is constant, see Fig. 1. In order to show strucuture we have limited the number of configurations used in the average so that some of the structure in the individual snapshots (field configurations) contributing to the average is still visible.

This needs to be contrasted to the behaviour of a solid, where we find that the snap-shots and the time averages show similar structures, but the fluctuations disappear in the averages, and we find a very clear solid structure, see Fig. 2. The identification of the crystalline structure can also be made by looking at the long-range order exhibited by the baryon-density correlation function

$$C(\vec{r}) = \Big\langle \int d^2r' B(\vec{r}')B(\vec{r}' - \vec{r}) \Big\rangle / B^2, \tag{4}$$

where B is the total baryon number. We have chosen to calculate the angle average $C(r)$, which should give all details of the correlations for a fluid, and contains partial information in the case of a solid. The behaviour for a solid is quite different from that of a liquid, as we can

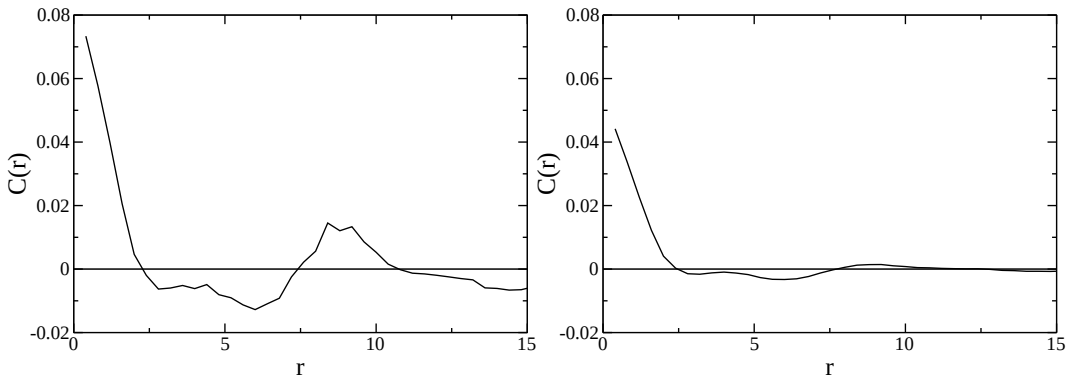

Fig. 3 – Correlation functions for a solid (left), and a liquid (right).

see in Fig. 3. The liquid shows some correlations, which is of course the distinction between liquid and gas, and shows that our identification is probably correct, even though we cannot be absolutely certain.

Unfortunately that is not the full story. We find no indication of a phase transition to a gas phase, but we do find something that can best be called phase coexistence, see Fig. 4. We find long solid-like networks (or, in other words, complicated multisolitons) and empty areas. We cannot be completely sure about their meaning, but the fact that this occurs for low densities may indicate some kind of droplet or grain formation.

We have performed numerous simulations, in order to get a more detailed picture of what is going on. The resulting phase diagram is shown in Fig. 5. We find that at finite temperatures a large part of the phase diagram is indeed dominated by a fluid phase, which gives us cause to investigate the traditional Skyrme model in more detail.

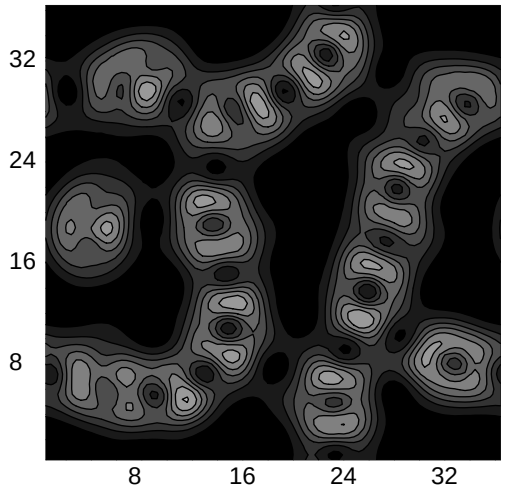

Fig. 4 – Ensemble averaged baryon density in the phase coexistence region.

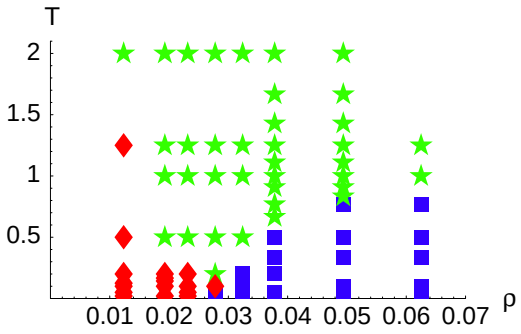

Fig. 5 – The phase diagram for the baby Skyrme model. Squares denote a solid, stars a liquid and lozenges phase coexistence.

Conclusions and Outlook. – We have not yet addressed the question what the nature of the nature of the phase transition is. Our results are not yet of sufficient quality to make a definite pronouncement on this issue. There are various indications that the phase transition is of low, maybe even first order. We have considered the energy as a function of ρ and T. We find indications of a sudden jump in flucutuations of the baryon density, see Fig. 6, at the phase transition, and a rapid change in energy. We are unable to state whether the change in the energy has a discontinuous slope; that will have to be investigated seperately. We also found an indication of latency (i.e., the point where the phase transition occurs depends on the direction in which we approach it). This usually indicates a first order phase transition, but the question is worthy of further investigation.

Of more immediate interest is the application to more realistic models, such as the 3D Skyrme model, or the QHE Skyrmions. Both of these are much more computationally intense problems, but we are currently making some inroads into both of these areas, using techniques similar to the ones reported here. We expect to be able to present some results in the near future [13, 14].

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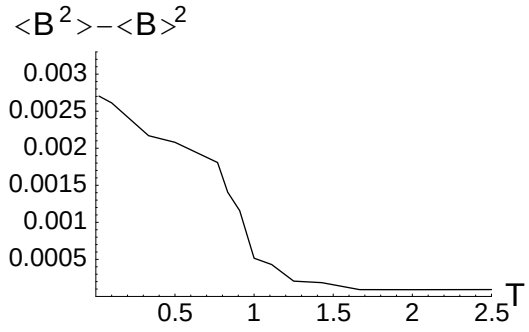

Fig. 6 – The fluctuations in baryon density for fixed ρ.

This work was supported by research grants (GR/L22331 and GR/N15672) from the Engineering and Physical Sciences Research Council (EPSRC) of Great Britain. OS has also been supported by an EPSRC studentship.

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